2 Distributions

Exercise 2.1. Let $\varphi \in \mathscr{S}(\mathbb{R})$. There are many ways to see that, but using that $(\log|x|)' = \text{p.v.}\dfrac{1}{x}$, we deduce that

$$\left\langle \text{p.v.}\frac{1}{x}, \varphi \right\rangle = -\int_{\mathbb{R}} \log|x|\varphi'(x)dx.$$

We have

$$\left|\int_{\mathbb{R}\setminus[-1,1]} \log|x|\varphi'(x)dx\right| \leq \int_{\mathbb{R}\setminus[-1,1]} |x||\varphi'(x)|dx \leq \sup_{x\in\mathbb{R}}|x^3\varphi'(x)|\int_{\mathbb{R}\setminus[-1,1]}\frac{dx}{|x|^3} = \|\varphi\|_{1,3}.$$

On the other hand, we have

$$\left|\int_{[-1,1]} \log|x|\varphi'(x)dx\right| \leq \|\log|x|\|_{\mathrm{L}^1([-1,1])}\|\varphi'\|_{\mathrm{L}^\infty(\mathbb{R})} = 2\|\varphi\|_{1,0}.$$

Finally, we get

$$\left|\left\langle \text{p.v.}\frac{1}{x}, \varphi \right\rangle\right| \leq (2\|\varphi\|_{1,0} + \|\varphi\|_{1,3}),$$

which shows (almost by definition) that $\text{p.v.}\dfrac{1}{x} \in \mathscr{S}'(\mathbb{R})$.

Exercise 2.2. Indeed, we have

$$\sup_{x\in\mathbb{R}}\left|x^\beta\partial_x^\alpha(\tau_a\varphi)(x)\right| = \sup_{x\in\mathbb{R}}\left|x^\beta\varphi^{(\alpha)}(x+a)\right|$$

Since $\varphi \in \mathscr{D}(\mathbb{R})$, if $\text{supp}(\varphi) \subset [-R,R]$, we deduce that

$$\begin{aligned}\sup_{x\in\mathbb{R}}\left|x^\beta\varphi^{(\alpha)}(x+a)\right| &= \sup_{|x+a|\leq R}\left|x^\beta\varphi^{(\alpha)}(x+a)\right| \\ &\leq \sup_{|x|\leq R+|a|}\left|x^\beta\varphi^{(\alpha)}(x+a)\right| \leq (R+|a|)^\beta\left\|\varphi^{(\alpha)}\right\|_{\mathrm{L}^\infty(\mathbb{R})} \leq C(1+|a|)^\beta.\end{aligned}$$

By contradiction, if $e^x \in \mathscr{S}'(\mathbb{R})$, there exists $n \in \mathbb{N}$ and $C < \infty$ such that

$$|\langle e^x, \varphi\rangle| \leq C\sup_{|\alpha|,|\beta|\leq n}\|\varphi\|_{\alpha,\beta}.$$

Then, we get

$$|\langle \tau_{-a}e^x, \varphi\rangle| \leq C_\varphi(1+|a|)^n,$$

whilst the identity $\tau_{-a}e^x = e^{x-a} = e^{-a}e^x$ shows that

$$|\langle e^x, \varphi\rangle| = e^a|\langle \tau_{-a}e^x, \varphi\rangle| \leq C_\varphi e^a(1+|a|)^n.$$

By letting $a \to -\infty$, we deduce that $\langle e^x, \varphi\rangle = 0$. Therefore, the only continuous extension of e^x to $\mathscr{S}'(\mathbb{R})$ is the 0 distribution, which shows that $e^x \notin \mathscr{S}'(\mathbb{R})$.

Exercise 2.3. For all $\varphi \in \mathscr{S}(\mathbb{R})$, we have

$$T(\varphi) = \sum_{n\in\mathbb{Z}} a_n \varphi(n).$$

If $\{a_n\}_{n\in\mathbb{N}}$ has polynomial growth, we directly estimate

$$\begin{aligned}|T(\varphi)| &\leq C\sum_{n\in\mathbb{N}}(1+|n|)^N\varphi(n) \leq C\left\|(1+|x|)^{N+2}\varphi\right\|_{\mathrm{L}^\infty(\mathbb{R})}\sum_{n\in\mathbb{Z}}\frac{1}{1+n^2}\\ &= C'\left\|(1+|x|)^{N+2}\varphi\right\|_{\mathrm{L}^\infty(\mathbb{R})} \leq C''\left(\|\varphi\|_{0,0}+\|\varphi\|_{N,0}\right).\end{aligned}$$

On the other hand, if $\{a_n\}_{n\in\mathbb{N}}$ does not have polynomial growth. By contradiction, assume that

$$|T(\varphi)| \leq C \sup_{|\alpha|,|\beta|\leq N} \|\varphi\|_{\alpha,\beta}.$$

Then, for all $m \in \mathbb{Z}$, we have

$$\left|\sum_{n\in\mathbb{Z}} a_{n+m}\varphi(n)\right| = |\langle \tau_m T, \varphi\rangle| = |\langle T, \tau_{-m}\varphi\rangle| \leq C_\varphi(1+|m|)^N.$$

Choosing $\varphi \in \mathscr{D}(\mathbb{R})$ such that $\varphi(0) = 1$ and $\mathrm{supp}(\varphi) \subset (-1,1)$, we deduce that

$$|a_m| = |\tau_m T(\varphi)| \leq C_\varphi(1+|m|)^N,$$

which contradicts the assumption that $\{a_n\}_{n\in\mathbb{N}}$ does not have polynomial growth at infinity.

Exercise 2.4. Using the structure theorem $T = \sum_{|\alpha|\leq m} D^\alpha \mu_\alpha$, where each μ_α has compact support, we need only check the property for a Radon measure $T = \mu$. *A priori*, we only have $T * \varphi \in \mathscr{D}'(\mathbb{R}^d)$. Furthermore, for all $\psi \in \mathscr{D}(\mathbb{R})$, we have

$$\begin{aligned}\langle T*\varphi, \psi\rangle &= \left\langle T_x, \int_{\mathbb{R}^d}\varphi(y)\psi(x+y)dy\right\rangle = \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\varphi(y)\psi(x+y)dy\right)d\mu(x)\\ &= \int_{\mathbb{R}^d}\psi(y)\left(\int_{\mathbb{R}^d}\varphi(y-x)d\mu(x)\right)dy,\end{aligned}$$

and the function

$$y \mapsto \int_{\mathbb{R}^d}\varphi(y-x)d\mu(x)$$

is well-defined since μ has compact support, and standard derivation under the integral estimates show that for all $\alpha \in \mathbb{N}^d$, we have

$$D^\alpha(T*\varphi)(x) = T * D^\alpha\varphi(x) = \int_{\mathbb{R}^d} D^\alpha\varphi(x-y)d\mu(y).$$

Therefore, we have $T * \varphi \in C^\infty(\mathbb{R}^d)$. Furthermore, if $\mathrm{supp}(\mu) \subset B(0,R)$, we deduce that

$$x^\beta D^\alpha(T*\varphi)(x) = \int_{B(0,R)} x^\beta D^\alpha \varphi(x-y)d\mu(y).$$

Now, notice that for all $y \in B(0,R)$, and for all $1 \leq j \leq d$, we have

$$|x_j|^{\beta_j} \leq 2^{\beta_j - 1}\left(|x_j - y_j|^{\beta_j} + |y_j|^{\beta_j}\right) \leq 2^{|\beta|-1}\left(|x_j-y_j|^{\beta_j} + R^{|\beta|}\right).$$

Therefore, we have

$$\begin{aligned}\left|x^\beta D^\alpha(T*\varphi)(x)\right| &\leq 2^{|\beta|-1}\int_{B(0,R)}\prod_{j=1}^d\left(|x_j-y_j|^{\beta_j}+R^{|\beta|}\right)|D^\alpha\varphi(x-y)|d\mu(y)\\ &\leq 2^{|\beta|-1}\left(1+R^{d|\beta|}\right)\sum_{\beta'\leq\beta}\int_{B(0,R)}|(x-y)^{\beta'}D^\alpha\varphi(x-y)|d\mu(y)\\ &\leq 2^{|\beta|-1}\left(1+R^{d|\beta|}\right)\mu(B(0,R))\sum_{\beta'\leq\beta}\|\varphi\|_{\alpha,\beta'}.\end{aligned}$$

We obtain the estimate

$$\|T*\varphi\|_{\alpha,\beta} \leq 2^{|\beta|-1}\left(1+R^{d|\beta|}\right)\mu(B(0,R))\sum_{\beta'\leq\beta}\|\varphi\|_{\alpha,\beta'}.$$

The last assertion follows immediately from this estimate. Let $C<\infty$ and $N \in \mathbb{N}$ such that for all $\varphi \in \mathscr{S}(\mathbb{R}^d)$,

$$|S(\varphi)| \leq C \sup_{|\alpha|,|\beta|\leq N}\|\varphi\|_{\alpha,\beta}.$$

Then, for all $x \in \mathbb{R}^d$, the function

$$F(x) = \langle S_y, \varphi(x+y)\rangle$$

satisfies

$$|F(x)| \leq C\sup_{|\alpha|,|\beta|\leq N}\left\|y^\beta D_y^\alpha\varphi(x+y)\right\|_{\mathrm{L}^\infty(\mathbb{R}^d)}.$$

We easily deduce from this identity that $F \in C^\infty(\mathbb{R}^d)$, and that

$$\|F\|_{\mathrm{L}^\infty(B(0,R))} \leq C'\sup_{|\alpha|,|\beta|\leq N}\|\varphi\|_{\alpha,\beta}$$

Finally, we deduce that

$$|\langle T*S,\varphi\rangle| = |T(F)| \leq C'\mu(B(0,R))\sup_{|\alpha|,|\beta|\leq N}\|\varphi\|_{\alpha,\beta},$$

where C' only depends on $R>0$.

Exercise 2.5. See the lecture notes for a general proof of this identity.